

On an identity of Ky Fan

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Abstract

We give several applications of an identity for sums of weakly stationary sequences due to Ky Fan.

1 Introduction and results

Let X_1, X_2, \dots be a weakly stationary sequence in an Hilbert H . In [6] (see p.598), Ky Fan observed that for any two positive integers n, m , one has

$$\begin{aligned} \frac{\|X_1 + \dots + X_n\|^2}{n} + \frac{\|X_1 + \dots + X_m\|^2}{m} - \frac{\|X_1 + \dots + X_{n+m}\|^2}{n+m} \\ = \frac{n(n+m)}{m} \left\| \frac{X_1 + \dots + X_n}{n} - \frac{X_1 + \dots + X_{n+m}}{n+m} \right\|^2. \end{aligned} \quad (1)$$

This nice identity was applied in the same paper. Rather surprisingly, this result did not seem to have caught much attention.

The object of this short note is to indicate some other consequences of identity (1), which have not been quoted in [5] or [6]. Before going further, and since no proof of this identity is given in [6], we thought worth to give one. The proof goes as follows. Put for any positive integer n , $S_n = X_1 + \dots + X_n$, and if m is another positive integer let $T_{n,m} = S_{n+m} - S_n$, so that $S_{n+m} = S_n + T_{n,m}$. Then

$$\left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2 = \frac{\|S_n\|^2}{n^2} + \frac{\|S_{n+m}\|^2}{(n+m)^2} - \left\langle \frac{S_n}{n}, \frac{S_{n+m}}{n+m} \right\rangle - \left\langle \frac{S_{n+m}}{n+m}, \frac{S_n}{n} \right\rangle,$$

and so

$$\begin{aligned} \frac{n(n+m)}{m} \left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2 \\ = \frac{(n+m)\|S_n\|^2}{nm} + \frac{n\|S_{n+m}\|^2}{m(n+m)} - \frac{1}{m} \{ \langle S_n, S_{n+m} \rangle + \langle S_{n+m}, S_n \rangle \} \\ = \frac{\|S_n\|^2}{n} + \left(\frac{\|S_n\|^2}{m} - \frac{\|S_m\|^2}{m} \right) + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m} \\ + \frac{\|S_{n+m}\|^2}{m} - \frac{1}{m} \{ \langle S_n, S_{n+m} \rangle + \langle S_{n+m}, S_n \rangle \}. \end{aligned}$$

But

$$\langle S_n, S_{n+m} \rangle + \langle S_{n+m}, S_n \rangle = 2\|S_n\|^2 + \langle S_n, T_{n,m} \rangle + \langle T_{n,m}, S_n \rangle,$$

so that in turn

$$\begin{aligned}
\frac{n(n+m)}{m} \left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2 &= \frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m} \\
&\quad + \frac{1}{m} \left(\|S_{n+m}\|^2 - \|S_n\|^2 - \|S_m\|^2 - \langle S_n, T_{n,m} \rangle - \langle T_{n,m}, S_n \rangle \right) \\
&= \frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m}, \tag{2}
\end{aligned}$$

since $S_{n+m} = S_n + T_{n,m}$. And we are done.

Note that the weak stationarity assumption was only used in the last line of calculations, to say that $\|T_{n,m}\| = \|S_m\|$. And consequently, if X_1, X_2, \dots is any sequence in H satisfying

$$\left\| \sum_{k=1}^m X_k \right\| \geq \left\| \sum_{k=n+1}^{n+m} X_k \right\|, \quad (n \geq 1, m \geq 1), \tag{3}$$

then, for any positive integers n, m

$$\frac{n(n+m)}{m} \left\| \frac{S_n}{n} - \frac{S_{n+m}}{n+m} \right\|^2 \leq \frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m} - \frac{\|S_{n+m}\|^2}{n+m}. \tag{4}$$

So is the case when for instance $X_i = T^i X_0$, $i = 1, \dots$ where T is a contraction in H , and X_0 some fixed element of H . A natural question concerns the possibility to replace the norming factor n^{-1} by another one. The lemma below shows that (4) remains true with norming factor $n^{-\delta}$, $1/2 < \delta \leq 1$.

Lemma 1 *Let $\{\alpha_k, k \geq 1\}$ be a sequence of positive reals satisfying the following condition:*

$$\alpha_{n+m} \leq \alpha_n + \alpha_m, \quad n, m \geq 1. \tag{5}$$

Let X_1, X_2, \dots be a sequence in H satisfying (3). Then, for any positive integers n, m

$$\frac{\alpha_n \alpha_{n+m}}{\alpha_m} \left\| \frac{S_n}{\alpha_n} - \frac{S_{n+m}}{\alpha_{n+m}} \right\|^2 \leq \frac{\|S_n\|^2}{\alpha_n} + \frac{\|S_m\|^2}{\alpha_m} - \frac{\|S_{n+m}\|^2}{\alpha_{n+m}} \left(\frac{\alpha_{n+m} - \alpha_n}{\alpha_m} \right). \tag{6}$$

Remark 2 — A typical case where Lemma 1 applies is when $\alpha_n = n^\delta$ with $0 \leq \delta \leq 1$, and we get in particular for any $m \geq n \geq 1$

$$\frac{n^\delta (n+m)^\delta}{m^\delta} \left\| \frac{S_n}{n^\delta} - \frac{S_{n+m}}{(n+m)^\delta} \right\|^2 \leq \frac{\|S_n\|^2}{n^\delta} + \frac{\|S_m\|^2}{m^\delta} - \frac{\|S_{n+m}\|^2}{(n+m)^\delta} \left(\frac{(n+m)^\delta - n^\delta}{m^\delta} \right). \tag{7}$$

— Notice also that (7) with $\delta = 1$ is (4).

Proof. Similarly as before

$$\frac{\alpha_n \alpha_{n+m}}{\alpha_m} \left\| \frac{S_n}{\alpha_n} - \frac{S_{n+m}}{\alpha_{n+m}} \right\|^2 = \frac{\|S_n\|^2}{\alpha_n} + \frac{\|S_m\|^2}{\alpha_m} - \frac{\|S_{n+m}\|^2}{\alpha_{n+m}}$$

$$\begin{aligned}
& + \frac{\|S_{n+m}\|^2}{\alpha_m \alpha_{n+m}} (\alpha_n - (\alpha_{n+m} - \alpha_m)) \\
& + \|S_n\|^2 \left(\frac{\alpha_{n+m} - (\alpha_n + \alpha_m)}{\alpha_n \alpha_m} \right) \\
& + \frac{1}{\alpha_m} (\|T_{n,m}\|^2 - \|S_m\|^2). \tag{8}
\end{aligned}$$

Using assumptions (3) and (5) gives

$$\frac{\alpha_n \alpha_{n+m}}{\alpha_m} \left\| \frac{S_n}{\alpha_n} - \frac{S_{n+m}}{\alpha_{n+m}} \right\|^2 \leq \frac{\|S_n\|^2}{\alpha_n} + \frac{\|S_m\|^2}{\alpha_m} - \frac{\|S_{n+m}\|^2}{\alpha_{n+m}} \left(\frac{\alpha_{n+m} - \alpha_n}{\alpha_m} \right),$$

as claimed. \blacksquare

A simple although quite interesting consequence of Ky Fan's identity is

$$\frac{\|S_{n+m}\|^2}{n+m} \leq \frac{\|S_n\|^2}{n} + \frac{\|S_m\|^2}{m}, \tag{9}$$

which is valid for any two positive integers n, m . This is inequality (4.8) in [6]. Recall that a sequence $\{g_n, n \geq 1\}$ of real numbers is subadditive when

$$g_{n+m} \leq g_n + g_m.$$

Then we have the well-known lemma

Lemma 3 *If $\{g_n, n \geq 1\}$ is a subadditive sequence of real numbers, then g_n/n converges to $\inf_{n \geq 1} (g_n/n)$.*

Proof. Fix an arbitrary positive integer N and write $n = j_n N + r_n$ with $1 \leq r_n \leq N$. Clearly $\frac{j_n}{n} \rightarrow \frac{1}{N}$ as n tends to infinity. Further

$$\inf_{n \geq 1} \frac{g_n}{n} \leq \frac{g_n}{n} \leq \frac{g_{j_n N} + g_{r_n}}{n} \leq \frac{g_{j_n N}}{j_n N} + \frac{g_{r_n}}{n} \leq \frac{j_n g_N}{j_n N} + \frac{g_{r_n}}{n} = \frac{g_N}{N} + \frac{g_{r_n}}{n}.$$

Letting now n tend to infinity gives $\inf_{n \geq 1} \frac{g_n}{n} \leq \liminf_{n \rightarrow \infty} \frac{g_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{g_n}{n} \leq \frac{g_N}{N}$. As N was arbitrary, the lemma is proved. \blacksquare

We thus deduce from (2) and this lemma applied to $g_n := \frac{\|S_n\|^2}{n}$ that

$$\lim_{n \rightarrow \infty} \left\| \frac{S_n}{n} \right\| = \inf_{n \geq 1} \left\| \frac{S_n}{n} \right\|. \tag{10}$$

This is a remarkable direct consequence of Ky Fan's identity, which remains true for averages of contractions. If T is a contraction in H , in view of the Riesz's decomposition ([4], lemma 1.3 p.4), the orthogonal complement H_T^\perp of $H_T = \{g \in H : Tg = g\}$ coincides with the closure of the subspace spanned by $\{h - Th : h \in H\}$. Then it suffices to proceed by approximation. In the next proposition we examine the ratios

$$\frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\left(\frac{1}{n_k} - \frac{1}{n_{k+1}} \right)},$$

where $\mathcal{N} = \{n_k, k \geq 1\}$ is an increasing sequence of positive integers. Notice that in the orthonormal case, namely if X_1, X_2, \dots is an orthonormal sequence, then precisely $\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2 = \frac{1}{n_k} - \frac{1}{n_{k+1}}$.

Proposition 4 (a) Let $\mathcal{N} = \{n_k, k \geq 1\}$ be an increasing sequence of positive integers. Then

$$\limsup_{N \rightarrow \infty} \frac{1}{n_N} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\left(\frac{1}{n_k} - \frac{1}{n_{k+1}} \right)} \leq \limsup_{N \rightarrow \infty} \sup_{1 \leq k < N} \left| \frac{\|S_{n_{k+1}-n_k}\|^2}{(n_{k+1} - n_k)^2} - \frac{\|S_{n_N}\|^2}{n_N^2} \right|.$$

(b) Further, if $\lim_{k \rightarrow \infty} n_{k+1} - n_k = \infty$,

$$\lim_{N \rightarrow \infty} \frac{1}{n_N} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\left(\frac{1}{n_k} - \frac{1}{n_{k+1}} \right)} = 0.$$

(c) And

$$\lim_{N, a \rightarrow \infty} \frac{1}{Na} \sum_{k=1}^{N-1} \left\| k(S_{a(k+1)} - S_{ak}) - S_{ak} \right\|^2 = 0.$$

(d) Finally let $\mathcal{D} = \{D_j, j \geq 1\}$ be a chain: $D_j | D_{j+1}$ for every j . Then

$$\sum_{j=1}^{\infty} \frac{1}{D_{j+1}} \sum_{k=1}^{\frac{D_{j+1}}{D_j}-1} \frac{\left\| \frac{S_{kD_j}}{kD_j} - \frac{S_{(k+1)D_j}}{(k+1)D_j} \right\|^2}{\frac{1}{kD_j} - \frac{1}{(k+1)D_j}} = \left(\frac{\|S_{D_1}\|}{D_1} \right)^2 - \lim_{J \rightarrow \infty} \left(\frac{\|S_{D_{J+1}}\|}{D_{J+1}} \right)^2 < \infty.$$

Remark 5 — A sequence $\{a_n, n \geq 1\}$ of real numbers converges in density to 0, which we write $D - \lim_{n \rightarrow \infty} a_n = 0$, if there exists a subset \mathcal{J} of \mathbf{N} of density one, such that $\lim_{\mathcal{J} \ni n \rightarrow \infty} a_n = 0$. For *bounded* sequences, it is an exercise to show that $D - \lim_{n \rightarrow \infty} a_n = 0$, if and only if, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} |a_n| = 0$. Since

$$\frac{\left\| \frac{S_{ak}}{ak} - \frac{S_{a(k+1)}}{a(k+1)} \right\|^2}{\frac{1}{ak} - \frac{1}{a(k+1)}} = \left\| k(S_{a(k+1)} - S_{ak}) - S_{ak} \right\|^2,$$

(c) means

$$\lim_{N, a \rightarrow \infty} \frac{1}{Na} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{ak}}{ak} - \frac{S_{a(k+1)}}{a(k+1)} \right\|^2}{\frac{1}{ak} - \frac{1}{a(k+1)}} = 0.$$

Thus along linearly growing sequences, the averages of weakly stationary sequences have, in density, increments comparable to averages of orthogonal sequences, which is a bit unexpected.

Proof. — From Ky Fan's identity, we get for each k

$$\frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\frac{1}{n_k} - \frac{1}{n_{k+1}}} = \left(\frac{\|S_{n_k}\|^2}{n_k} - \frac{\|S_{n_{k+1}}\|^2}{n_{k+1}} \right) + \frac{\|S_{n_{k+1}-n_k}\|^2}{n_{k+1} - n_k}.$$

Summing from $k = 1$ up to $N - 1$ leads to

$$\sum_{k=1}^{N-1} \frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\frac{1}{n_k} - \frac{1}{n_{k+1}}} = \frac{\|S_{n_1}\|^2}{n_1} - \frac{\|S_{n_N}\|^2}{n_N} + \sum_{k=1}^{N-1} (n_{k+1} - n_k) \left(\frac{\|S_{n_{k+1}-n_k}\|}{n_{k+1} - n_k} \right)^2 \quad (11)$$

Dividing both sides by n_N gives

$$\begin{aligned}
\frac{1}{n_N} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\frac{1}{n_k} - \frac{1}{n_{k+1}}} &= \frac{\|S_{n_1}\|^2}{n_1 n_N} - \frac{\|S_{n_N}\|^2}{n_N^2} \\
&\quad + \frac{1}{n_N} \sum_{k=1}^{N-1} (n_{k+1} - n_k) \left(\frac{\|S_{n_{k+1}-n_k}\|}{n_{k+1} - n_k} \right)^2 \\
&\leq \frac{\|S_{n_1}\|^2}{n_1 n_N} + \frac{1}{n_N} \sum_{k=1}^{N-1} (n_{k+1} - n_k) \left\{ \left(\frac{\|S_{n_{k+1}-n_k}\|}{n_{k+1} - n_k} \right)^2 \right. \\
&\quad \left. - \frac{\|S_{n_N}\|^2}{n_N^2} \right\}. \tag{12}
\end{aligned}$$

Letting next N tend to infinity yields

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{1}{n_N} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\left(\frac{1}{n_k} - \frac{1}{n_{k+1}} \right)} &\leq \limsup_{N \rightarrow \infty} \frac{1}{n_N} \sum_{k=1}^{N-1} (n_{k+1} - n_k) \left\{ \left(\frac{\|S_{n_{k+1}-n_k}\|}{n_{k+1} - n_k} \right)^2 - \frac{\|S_{n_N}\|^2}{n_N^2} \right\} \\
&\leq \limsup_{N \rightarrow \infty} \sup_{1 \leq k < N} \left| \frac{\|S_{n_{k+1}-n_k}\|^2}{(n_{k+1} - n_k)^2} - \frac{\|S_{n_N}\|^2}{n_N^2} \right|. \tag{13}
\end{aligned}$$

Hence the first claim is proved.

— If $\lim_{k \rightarrow \infty} n_{k+1} - n_k = \infty$ and suppose first that $\lim_{n \rightarrow \infty} \frac{\|S_n\|}{n} = 0$. Then $\lim_{k \rightarrow \infty} \frac{\|S_{n_{k+1}-n_k}\|}{n_{k+1}-n_k} = 0$, and so letting N tend to infinity in the first equality in (11) gives

$$\lim_{N \rightarrow \infty} \frac{1}{n_N} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\left(\frac{1}{n_k} - \frac{1}{n_{k+1}} \right)} = 0.$$

Hence the second claim of the proposition is proved in that case. If $\lim_{n \rightarrow \infty} \frac{\|S_n\|}{n} > 0$, there exists $\chi \in H$ such that $\lim_{n \rightarrow \infty} \left\| \frac{S_n}{n} - \chi \right\| = 0$. Indeed, first recall ([4], p.32) that $\{X_i, i \geq 1\}$ may be represented as a sequence $\{T^i X_1, i \geq 0\}$ where T is an isometry in some Hilbert space, which we denote again H . Next by the mean ergodic theorem of von Neumann ([4], p.4), the limit χ is identified as the projection on the subspace $H_T = \{g \in H : Tg = g\}$ of X_1 . Applying the result previously obtained to the weakly stationary sequence $\{X_i - \chi, i \geq 1\}$, allows to reach the same conclusion in this case as well.

— Now assume $n_k = ak$, a being some fixed positive integer. Replace n_k by its value in the first part of (11).

$$\begin{aligned}
\frac{1}{Na} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{ka}}{ka} - \frac{S_{(k+1)a}}{(k+1)a} \right\|^2}{\left(\frac{1}{ka} - \frac{1}{(k+1)a} \right)} &= \frac{\|S_a\|^2}{a^2 N} - \frac{\|S_{aN}\|^2}{a^2 N^2} + \frac{1}{N} \sum_{k=1}^{N-1} \frac{\|S_a\|^2}{a^2} \\
&= \frac{\|S_a\|^2}{a^2} - \frac{\|S_{aN}\|^2}{a^2 N^2} \tag{14}
\end{aligned}$$

Hence

$$\limsup_{N,a \rightarrow \infty} \frac{1}{Na} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{ka}}{ka} - \frac{S_{(k+1)a}}{(k+1)a} \right\|^2}{\left(\frac{1}{ka} - \frac{1}{(k+1)a} \right)} \leq \limsup_{N,a \rightarrow \infty} \left| \frac{\|S_a\|^2}{a^2} - \frac{\|S_{Na}\|^2}{(Na)^2} \right| = 0.$$

The expression in the right-hand side being also rewritten as

$$\frac{1}{Na} \sum_{k=1}^{N-1} \left\| k(S_{a(k+1)} - S_{ak}) - S_{ak} \right\|^2,$$

we get (c).

— Let $\{D_j, j \geq 1\}$ be a chain, and applies (12) with $N = D_{j+1}/D_j$, $a = D_j$.

$$\frac{1}{Na} \sum_{k=1}^{N-1} \frac{\left\| \frac{S_{ka}}{ka} - \frac{S_{(k+1)a}}{(k+1)a} \right\|^2}{\left(\frac{1}{ka} - \frac{1}{(k+1)a} \right)} = \frac{\|S_a\|^2}{a^2} - \frac{\|S_{aN}\|^2}{a^2 N^2} \quad (15)$$

We obtain

$$\frac{1}{D_{j+1}} \sum_{k=1}^{\frac{D_{j+1}}{D_j}-1} \frac{\left\| \frac{S_{kD_j}}{kD_j} - \frac{S_{(k+1)D_j}}{(k+1)D_j} \right\|^2}{\frac{1}{kD_j} - \frac{1}{(k+1)D_j}} = \left(\frac{\|S_{D_j}\|}{D_j} \right)^2 - \left(\frac{\|S_{D_{j+1}}\|}{D_{j+1}} \right)^2.$$

Summing up from $j = 1$ to $j = J$ gives

$$\sum_{j=1}^J \frac{1}{D_{j+1}} \sum_{k=1}^{\frac{D_{j+1}}{D_j}-1} \frac{\left\| \frac{S_{kD_j}}{kD_j} - \frac{S_{(k+1)D_j}}{(k+1)D_j} \right\|^2}{\frac{1}{kD_j} - \frac{1}{(k+1)D_j}} = \left(\frac{\|S_{D_1}\|}{D_1} \right)^2 - \left(\frac{\|S_{D_{J+1}}\|}{D_{J+1}} \right)^2.$$

Therefore

$$\sum_{j=1}^{\infty} \frac{1}{D_{j+1}} \sum_{k=1}^{\frac{D_{j+1}}{D_j}-1} \frac{\left\| \frac{S_{kD_j}}{kD_j} - \frac{S_{(k+1)D_j}}{(k+1)D_j} \right\|^2}{\frac{1}{kD_j} - \frac{1}{(k+1)D_j}} = \left(\frac{\|S_{D_1}\|}{D_1} \right)^2 - \lim_{J \rightarrow \infty} \left(\frac{\|S_{D_{J+1}}\|}{D_{J+1}} \right)^2 < \infty.$$

Hence (d), the proof is now complete. ■

For arithmetic progressions, Proposition 4 shows that

$$\frac{\left\| \frac{S_{ka}}{ka} - \frac{S_{(k+1)a}}{(k+1)a} \right\|^2}{\left(\frac{1}{ka} - \frac{1}{(k+1)a} \right)}$$

”asymptotically” converges in density to 0. The question naturally arises when for *any* increasing sequence \mathcal{N} , a convergence in density to 0 do hold.

In our next result, we give an example having this property. Recall that a Dunford-Schwartz contraction is a linear contraction T on L_1 of a σ -finite measure space, with $\|Tf\|_{\infty} \leq \|f\|_{\infty}$ for $f \in L_1 \cap L_{\infty}$, and induces a contraction on all L_p , $1 < p \leq \infty$ ([3]; see also [4] p. 65 for T positive). The limit $E(T)f := \lim_n \frac{1}{n} \sum_{k=1}^n T^k f$ exists a.e. for $f \in L_p$, $1 \leq p < \infty$, and also in L_p -norm for $p > 1$ (and in L_1 -norm in probability spaces). Write $S_n^T = \sum_{l=1}^n T^l$.

Let $0 < \alpha < 1$. By Corollary 2.15 in [2], when T is induced on L_p by a Dunford-Schwartz operator, if $f \in (I - T)^\alpha L_p$, then

$$\lim_{n \rightarrow \infty} \left\| \frac{S_n^T f}{n^{1-\alpha}} \right\|_p = 0.$$

If T is an isometry on L_2 of a σ -finite measure space, for instance if T is the isometry induced by an ergodic transformation, then $\sqrt{I - T}L_2$ is a dense sub-space in the space of elements $f \in L_2$ such that $\langle f, 1 \rangle = 0$; and strictly contains the space of coboundaries. And we have the following characterization ([1] Proposition 2.2): $f \in \sqrt{I - T}L_2$ if and only if the series $\sum_{n=1}^{\infty} \|S_n^T f\|^2/n^2$ converges. A spectral characterization is further given in Proposition 2.2 of [1] (see also [2] Theorem 4.4).

Theorem 6 *Let T be an isometry on L_2 of a σ -finite measure space, and $f \in \sqrt{I - T}L_2$. Then for any increasing sequence $\mathcal{N} = \{n_k, k \geq 1\}$ of positive integers, we have*

$$\lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \frac{\left\| \frac{S_{n_k}^T f}{n_k} - \frac{S_{n_{k+1}}^T f}{n_{k+1}} \right\|^2}{\left(\frac{1}{n_k} - \frac{1}{n_{k+1}} \right)} = 0.$$

Proof. We apply the previous remark with $p = 2$, $\alpha = 1/2$. By (11)

$$\begin{aligned} \sum_{k=1}^{K-1} \frac{\left\| \frac{S_{n_k}}{n_k} - \frac{S_{n_{k+1}}}{n_{k+1}} \right\|^2}{\frac{1}{n_k} - \frac{1}{n_{k+1}}} &= \frac{\|S_{n_1}\|^2}{n_1} - \frac{\|S_{n_K}\|^2}{n_K} + \sum_{k=1}^{K-1} \frac{\|S_{n_{k+1}-n_k}\|^2}{n_{k+1} - n_k} \\ &\leq \frac{\|S_{n_1}\|^2}{n_1} + o(K). \end{aligned}$$

Dividing both sides by K and letting next K tend to infinity achieves the proof. \blacksquare

We conclude with another inequality. Put

$$f^2(n) = \sup_{n' \leq n} \frac{\|S_{n'}\|^2}{n'}.$$

Lemma 7 *f^2 is subadditive: we have $f^2(n + m) \leq f^2(n) + f^2(m)$ for any $m, n \geq 1$.*

Proof. There is no loss to assume $n \leq m$. Let $\mu \leq m + n$. Consider three cases:
i) $\mu \leq n$. Then $\|S_\mu\|^2/\mu \leq f(n) \leq f(n) + f(m)$.

ii) $n < \mu \leq m$. Write $\mu = \mu - n + n$. We have $\mu - n \leq m$ and using (9)

$$\frac{\|S_\mu\|^2}{\mu} = \frac{\|S_{(\mu-n)+n}\|^2}{(\mu-n)+n} \leq \frac{\|S_{\mu-n}\|^2}{\mu-n} + \frac{\|S_n\|^2}{n} \leq f^2(\mu-n) + f^2(n) \leq f^2(m) + f^2(n).$$

iii) $m < \mu \leq n + m$. Then $\mu = \mu - m + m$. We have $\mu - m \leq n + m - m = n$ and using (9) again

$$\frac{\|S_\mu\|^2}{\mu} \leq \frac{\|S_{\mu-m}\|^2}{\mu-m} + \frac{\|S_m\|^2}{m} \leq f^2(\mu-m) + f^2(m) \leq f^2(n) + f^2(m).$$

\blacksquare

Put

$$I(f, x, y) = \frac{f^2(x+y) - f^2(x) + f^2(y)}{2f(x+y)f(y)}.$$

Observe that if $h(x) = |x|^{1/2}$, then $I(h, x, y) = 2\sqrt{\frac{y}{y+x}}$.

Proposition 8 *For any positive integers x, y such that $f^2(x)/x \geq f^2(y)/y$, we have*

$$I(f, x, y) \leq 2\sqrt{\frac{y}{y+x}}.$$

Proof. Write

$$f^2(x+y) = p(x+y), \quad f^2(y) = qy, \quad f^2(x) = rx.$$

Substituting these values into $I(f, x, y)$ gives

$$I(f, x, y) = \frac{ph^2(x+y) - rh^2(x) + qh^2(y)}{2\sqrt{pq}h(x+y)h(y)}$$

Consider the following expression

$$J = \left(I(f, x, y) - I(h, x, y) \right) 2h(x+y)h(y) = p(x+y) + qy - rx - 2y\sqrt{pq}$$

It suffices to prove $J \leq 0$. By lemma 3, $p(x+y) \leq rx + qy$.

— If $q \leq p$, then $J \leq 2qy - 2y\sqrt{pq} = 2y\sqrt{q}[\sqrt{q} - \sqrt{p}] \leq 0$. — If $q \geq p$, then

$$J = -\sqrt{p}(\sqrt{q} - \sqrt{p})(x+y) + \sqrt{q}(\sqrt{q} - \sqrt{p})y + (\sqrt{pq} - r)x.$$

We have $\sqrt{q} - \sqrt{p} \geq 0$, moreover $qy = f^2(y) \leq f^2(x+y) = p(x+y)$. Thus

$$\begin{aligned} J &\leq \left[-\sqrt{p}(\sqrt{q} - \sqrt{p})\frac{q}{p} + \sqrt{q}(\sqrt{q} - \sqrt{p}) \right] y + (\sqrt{pq} - r)x \\ &= -\frac{\sqrt{q}}{\sqrt{p}}(\sqrt{q} - \sqrt{p})^2 y + (\sqrt{pq} - r)x. \end{aligned}$$

But we assumed $f^2(x)/x \geq f^2(y)/y$, thus $r \geq q \geq p$. And so $\sqrt{pq} - r \leq 0$. Therefore $J \leq 0$ as required. \blacksquare

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